Pigeonhole Principle and Double Counting

arisu **Rizqi**

1. Introduction

In this article, we will discuss two of the most fundamental principles in combinatorics, Pigeonhole Principle and Double Counting. The pigeonhole principle will be introduced from the basic theorem, and then followed by surprising applications in various mathematical objects, specifically in number theory, and sequences of real numbers. Next, we will discuss Double Counting, which is based on the idea of counting the same set twice in different ways to give an equality. The main application of this technique is the Handshake Lemma in Graph Theory and its usage in the proof of Sperner's Lemma.

2. Pigeonhole Principle

Pigeonhole Principle is a fundamental principle in combinatorics. Even though it looks simple, this principle has broad applications in mathematics.

2.1. Theorem Statement

Theorem 2.1.1 (Pigeonhole Principle)

If n objects are placed in r boxes, where $r < n$, then at least one of the boxes contains more than one object.

Proof. Suppose to the contrary that all of the boxes contains at most one object, then $r \geq n$ which is a contradiction. □

In the language of functions, the above theorem can be rewritten and be made stronger as follows.

Theorem 2.1.2 (Pigeonhole Principle var.)

Let N and R be two finite sets with $|N| > |R|$, and $f : N \to R$. Then, there exists some $a \in R$ such that $|f^{-1}(a)| \geq 2$. In fact, $|f^{-1}(a)| \geq \lceil \frac{n}{r} \rceil$.

Next, we will show some direct applications of the Pigeonhole Principle.

Example.

1. In a class with 37 students, there must be at least 4 students who has the same birth month.
2. Let $A = \{1, 2, 3, ..., 16, 17\}$. If we take 9 numbers from A such that the sum of any two numbers taken is not 18, then 9 must be one of the numbers taken from A.

2.2. Number Theory

Other than the simple examples above, PHP can also be used to prove some facts in Number Theory.

Proposition 2.2.1

Consider the numbers 1, 2, 3, ..., 2n, and take $n + 1$ of them. Then, there are two numbers which are relatively prime.

Proof. Send each number m to a box with the label $\lceil \frac{m}{2} \rceil$. Then, there must be at most n boxes, but there are $n + 1$ numbers taken. Therefore, from PHP there must be two numbers in the same box, and the difference of those numbers is one from the rule above, so the two numbers must be relatively prime. □

Proposition 2.2.2

Consider the numbers 1, 2, 3, ..., 2n, and take $n + 1$ of them. Then, there are two numbers such that one divides

Proof. Write $a = 2^k m$ with m odd. Send each number a to the box with the label m. Because there are only n odd numbers, then there must be n boxes. Therefore, by PHP, there exists two numbers with the same value of m. Thus, the smaller number will divide the bigger one. □

2.3. Sequences

Other than that, PHP can be also used to prove theorems regarding sequences of real numbers.

Proposition 2.3.1 (Erdős–Szekeres)

In any sequence of $mn + 1$ distinct real numbers, there exist an increasing subsequence of length $m + 1$, decreasing subsequence of length $n + 1$, or both.

Proof. Send each number a_i to t_i, the length of the longest increasing subsequence starting at a_i. If $t_i \geq m + 1$ for some i then we are done. Otherwise, we have $t_i \leq m$ for all i. So, the range of the previous mapping has size m but we have $mn + 1$ numbers. By PHP, we have $\left\lceil \frac{mn+1}{m} \right\rceil = n + 1$ numbers such that $t_i = s$ for some $s \leq m$. Now for each consecutive number a_{j_i} and $a_{j_{i+1}}$, it must be the case that $a_{j_i} > a_{j_{i+1}}$ for otherwise we would have an increasing sequence of length $s + 1$ starting from a_{j_i}. Therefore we have a decreasing subsequence of length $n + 1$. □

2.4. Application of Erdős–Szekeres

Now, we will see some applications of the Erdős–Szekeres Theorem.

Theorem 2.4.1

Let K_n be a complete simple graph on n vertices. Define $\dim(K_n)$ as the minimum numbers of permutations $\Pi = \{\pi_1, \pi_2, ..., \pi_m\}$ on the set $\{1, 2, ..., n\}$ that represents K_n. A set of permutations is said to *represent* K_n if for each three distinct numbers $i, j, k \in \{1, 2, ..., n\}$, there exist a permutation $\pi \in \Pi$ in which k appears after *both* i and j in the permutation. Then, $\dim(K_n) \geq \log_2(\log_2 n)$

Proof. We need to show that $\dim(K_n) \geq p + 1$, for $n = 2^{2^p} + 1$. Suppose to the contrary that dim $(K_n) \leq p$, and let $\pi_1, \pi_2, ..., \pi_p$ be representing permutations of $N = \{1, 2, ..., 2^{2^p} + 1\}$. In the first permutation π_1, there exist a monotone subsequence A_1 yang with length $n_1 = 2^{2^{p-1}} + 1$. Then, we can find another monotone subsequence A_2 of A_1 in π_2 with length $n_2 = 2^{2^{p-2}} + 1$ which is also monotone in π_1. In the end, we will get a subsequence A_p with length $2^{2^0} + 1 = 3$. Let $A_p = (a, b, c)$. This subsequence is monotone in all $\pi_1, \pi_2, ..., \pi_p$. Therefore, the ordering must be $a < b < c$ or $a > b > c$. However, this is impossible since there must be a permutation where b appears after both a and c, which is a contradiction. □

3. Double Counting

Other than PHP, there also exist another fundamental technique in combinatorics, that is Double Counting. The idea is to count the same object twice using two different ways so that we can have an equality.

3.1. Theorem Statement

Theorem 3.1.1 (Double Counting)

Let R and C be finite sets and $S \subseteq R \times C$. The elements p and q are said to be *incident* if $(p, q) \in S$. Let r_p be the number of elements incident to $p \in R$ and c_q be the number of elements incident to $q \in C$. Then,

$$\sum_{p \in R} r_p = |S| = \sum_{q \in C} c_q$$

Proof. If we write the elements of S in a grid, this is the same as counting by rows or by columns which will give you the same thing. □

3.2. Graph Theory

One of the most famous application of this technique is the Handshake Lemma from Graph Theory.

Lemma 3.2.1 (Handshake Lemma)

Let $G = (V, E)$ be a simple graph, and $d(v)$ denote the degree of the vertex v. Then,

$$\sum_{v \in V} d(v) = 2|E|$$

Proof. Consider $S \subseteq V \times E$ where $(v, e) \in S$ iff v is an end-vertex of e. Counting in one way gives us $\sum_{v \in V} d(v)$ while counting in the other way gives us $2|E|$ since every edge has exactly two end-vertices. □

3.3. Sperner's Lemma

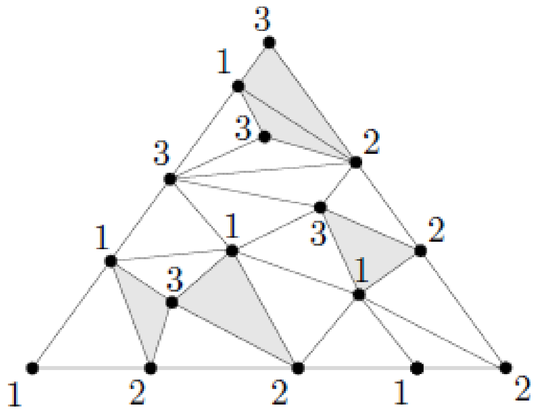
Lastly, we will discuss a lemma that uses double counting in its proof, but can be used to prove a theorem that is more difficult to prove directly without the lemma.

Lemma 3.3.1 (Sperner's Lemma)

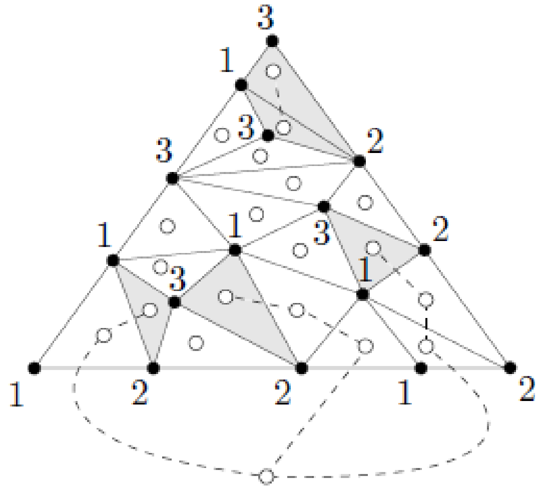
Suppose we have a triangle with vertices V_1, V_2, V_3 that is triangulated (decomposed into a finite number of smaller triangles). Color each vertex with 3 colors such that V_i gets the color i, and the edge along V_i and V_j only contains the color i and j. Then, there must be a smaller triangle in which the vertex has 3 different colors.

Proof. Consider the *partial* dual graph (faces become vertices and neighboring faces are connected by an edge) where we only take edges that crosses a 1-2 edge in the original graph. On this dual graph, there are only 3 possibilities for the degree of an internal vertex: 0 if the triangle does not contain color 1 or 2, 1 if the triangle is tricolored, and 2 if the triangle only has color 1 and 2.

However, the external vertex has odd degree since along the edge from V_1 to V_2 there are odd alternations between 1 and 2. Therefore, by the previous proposition there are an odd number of vertex of degree 1 which corresponds to tricolored triangles. □



Gambar 1: Graf awal



Gambar 2: Graf dual

It turns out that we can prove the fixed point theorem below using Sperner's Lemma. In this article, we will prove the theorem for $n = 2$.

Theorem 3.3.2 (Fixed Point Theorem)

Every continuous function $f : B^n \to B^n$ of an n-dimensional ball to itself has a fixed point, that is a point x where $f(x) = x$.

Proof. Define a triangle $\triangle$ in $\mathbb{R}^3$ with endpoints $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. It suffices to prove for continuous functions $f : \triangle \to \triangle$ because $\triangle$ is homeomorphic with B^2.

Define $\delta(T)$ as the maximum length of the edges in the triangulation of T. Construct a sequence of triangulations $T_1, T_2, \ldots$ of $\triangle$ such that $\delta(T_n) \to 0$ by adding vertices on each smaller triangles in di T_k to construct T_{k+1}.

For each triangulation T_n, define a coloring with $\lambda(v) = \min\{i : f(v)_i < v_i\}$, that is the smallest index such that the coordinate i of $f(v) - v$ is negative. Suppose to the contrary that f doesn't have a fixed point. We will show that the coloring is well defined. Notice that if $v \in \triangle$, then $v_1 + v_2 + v_3 = 1$, so, $f(v)_1 - v_1 + f(v)_2 - v_2 + f(v)_3 - v_3 = 0$ because $f(v) \in \triangle$. Because f doesn't have a fixed point, there must be an i such that $f(v)_i - v_i \neq 0$, so one must be negative and the other positive.

Now, we will check that the coloring fulfills the hypothesis of Sperner's Lemma. Every endpoint e_i, must be colored i because only at the i-th coordinate that $f(e_i)_i - e_{i_i} < 0$. Then, if v is on the edge that is opposite of e_i, we have $v_i = 0$ so it's impossible for v to be colored i.

Therefore, by Sperner's Lemma, we have that for each triangulation T_n, there exist a triangle $\{v^{k:1}, v^{k:2}, v^{k:3}\}$ that has 3 different colors with $\lambda(v^{k:i}) = i$. Because $\triangle$ is a compact set, the sequence $v^{k:1}$ has a subsequence that is convergent to v. Then, since $\delta(T_n) \to 0$, it must be the case that $v^{k:2}$ and $v^{k:3}$ convergent to the same point.

Now, notice that for every k, $f\left(v^{k:1}\right)_1 - v_1^{k:1} < 0$, it must be that $f(v)_1 - v_1 < 0$ since f is continuous. Similarly, we can prove that $f(v)_i - v_i < 0$ for $i = 2$ and $i = 3$. Therefore, there is no coordinate i such that $f(v)_i - v_i \geq 0$, and this contradicts with the assumption that f doesn't have a fixed point, since there must be one coordinate such that $f(v)_i - v_i > 0$ (and the other is negative). So, f must have a fixed point. $\square$

Bibliography

[1] M. Aigner and G. M. Ziegler, *Proofs from THE BOOK*. 2018. doi: 10.1007/978-3-662-57265-8.